Problem 2 – Kangaroo
Day 1 – Task 2

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Solution – 100 points

A sequence of jumps from cs to cf can be mirrored and thus obtaining a valid sequence of jumps from cf to cs. Due to this symmetry, cs and cf can be swapped such that cs < cf.

We will add the following definitions:

A[n][i][j] = the number of alternating permutations that start ascending, having the first and the last elements i and j
D[n][i][j] = the number of alternating permutations that start descending, having the first and the last elements i and j
X[n][i][j] = A[n][i][j] + D[n][i][j], (our target)
Y[n][i][j] = A[n][i][j] - D[n][i][j], (for formal reasons)

Let's consider an alternating permutation of length n that starts with i and ends with j. By removing the left extremity (i) and decreasing all values greater than i by 1, we'll obtain an alternating permutation of order n-1. We can infer the following recurrences:

```
A[n][i][j] = D[n-1][i][j-1] + D[n-1][i+1][j-1] +…+ D[n-1][n-2][j-1]
D[n][i][j] = A[n-1][1][j-1] + A[n-1][2][j-1] +…+ A[n-1][i-1][j-1],
```

In other words, the number of alternating permutations of length n starting ascending with i and ending with j is equal with the number of alternating permutations of length n-1 starting descending with i, i + 1, ..., n-1 and ending with j-1. Similarly for D[][][].

The above recurrences can be rewritten more conveniently:

```
A[n][i][j] = A[n][i-1][j] - D[n-1][i-1][j-1],
D[n][i][j] = D[n][i-1][j] + A[n-1][i-1][j-1],
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and from here we obtain

```
X[n][i][j] = X[n][i-1][j] + Y[n-1][i-1][j-1],
Y[n][i][j] = Y[n][i-1][j] - X[n-1][i-1][j-1]
```

After a few manipulations we can further derive:

X[n][i][j] = 2·X[n][i-1][j] - X[n][i-2][j] - X[n-2][i-2][j-2],
n>=3, i>=3

Let's stop for a moment to assess the complexity. The answer can be easily computed in **O(N^3)** using the above recurrence, but it can be reduced to **O(N^2)** as follows.

Note the following invariant that is preserved by the recurrence:

n - j = (n-2) – (j-2) = constant

This is a key observation that shows the first and the third index of X[][][] will not be independent of each other by repeatedly using the recurrence starting backwards from X[N][cs][cf]. Therefore, instead of three independent variables, (n, i, j), we’ll have only two (since N - cf = n – j = constant) so the complexity will be **O(N^2)** for a proper implementation.

We have one more thing to do, namely handling the corner cases i<=2:
Case n mod 2 = 1
X[n][1][j] = A[n][1][j] = A[n][j][1]
A[n][j][1] = D[n-1][j][1] + D[n-1][j+1][1] + … + D[n-1][n-1][1]
A[n][j][1] = A[n-1][1][j] + A[n-1][1][j+1] + … + A[n-1][1][n-1]
A[n][1][j] = A[n][1][j-1] - A[n-1][1][j-1]
Case n mod 2 = 0
X[n][1][j] = A[n][1][j] = D[n][j][1]
D[n][j][1] = A[n-1][j-1][1] + A[n-1][j-2][1] + … + A[n-1][3][1]
D[n][j][1] = A[n-1][1][j-1] + A[n-1][1][j-2] + … + A[n-1][1][3]
A[n][1][j] = A[n][1][j-1] + A[n-1][1][j-1]
resulting in
X[n][1][j] = X[n][1][j-1] - X[n-1][1][j-1], n mod 2 = 1
X[n][1][j] = X[n][1][j-1] + X[n-1][1][j-1], n mod 2 = 0

We have also:
A[n][2][j] = A[n][1][j] - D[n-1][1][j-1] = A[n][1][j] = X[n][1][j]
D[n][2][j] = D[n][1][j] + A[n-1][1][j-1] = A[n-1][1][j-1] = X[n-1][1][j-1]
(there is no descending permutation starting with 1)
X[n][2][j] = X[n][1][j] + X[n-1][1][j-1]