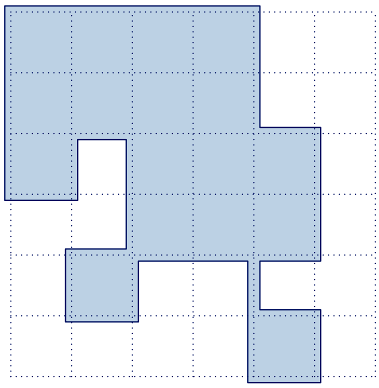
Solution

Full solution

Given a wall starting and ending in the upper left corner, we consider the set of points not reachable from outside the gird without crossing the wall and we will call this the *region* associated to the wall. Here, we assume the wall to be built some positive small amount left of the corresponding grid line (when facing the direction the wall was built). Thus, the region associated to a wall will always be connected and we want it to contain all the villages. Conversely, the boundary of any connected region without holes containing all villages is a valid wall and it is in fact a wall of minimum cost with that associated region.

Figure 1: A wall and its region



Fix some shortest path P_v from the top left corner of the grid to the top left corner of each village v (where the cost of a grid edge is the cost of building a wall along it).

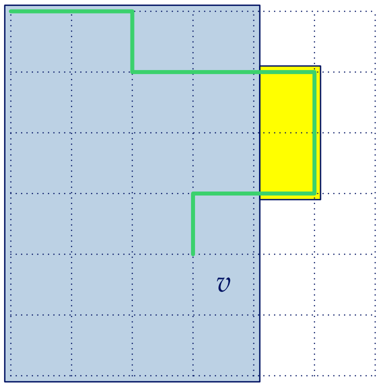
Lemma 1. *There is a wall of minimum cost protecting all the villages, the region associated to which contains all the shortest paths* P_v.

Proof. We will inductively enlarge the region R associated to any wall W protecting all the villages without increasing the cost of the wall.

Assume the region doesn't contain the path P_v to the village v. Consider the region $R' \supsetneq R$ of points not reachable from the bottom or from the right without crossing W or P_v. Let W' be the wall associated to R'. It is now sufficient to prove that the cost of W' is less than or equal to the cost of W. Both ends of P_v lie inside the region R since it contains the top left corner and the village v. The parts of P_v not contained in R are a sequence of subpaths of P_v not intersecting each other (since P_v doesn't intersect itself). Now, W' is obtained from W by replacing the parts of W connecting endpoints of such subpaths of P_v with the subpaths themselves. But since, by definition, all subpaths of P_v are cost-minimizing, the cost of W' is at most the cost of W. $\square$

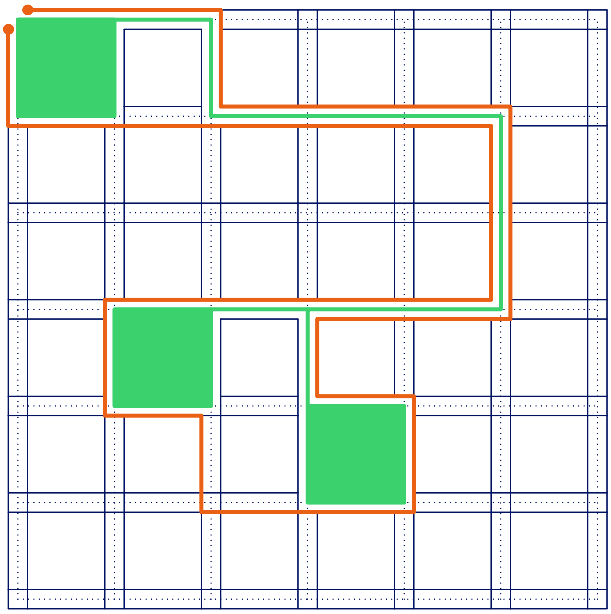
Let F be the union of the paths P_v and the villages v. We have reduced the problem to finding a wall of minimum cost enclosing F. But, since the region F is connected, we just have to find a cost-minimizing wall not crossing the border of F (and containing any

Figure 2: Proof of Lemma 1 (blue: R; blue and yellow: R'; green: P_v)



point of F, e.g. the city). This can be done in $\mathcal{O}(nm\log(nm))$ using Dijkstra's algorithm by splitting each node of the grid into four (at a small distance) and introducing an edge between two neighbouring of these four nodes if and only if their connection doesn't cross any path P_v and both nodes lie outside all villages.

Figure 3: Graph to run Dijkstra's algorithm on (green: F; dots: source and target; length of short edges: 0; length of long edges: cost of corresponding segment; orange: shortest path = minimum-cost wall)



Partial solution (25 points)

For any set S of villages and grid point (x,y) let $f(S,x,y)$ be the minimum cost of a wall starting at $(0,0)$ and ending at (x,y) that passed right of a village v an odd number of times if and only if $v \in S$. The correct output is then $f(V,0,0)$ if V is the set of all villages. But

$f(S,x,y)$ can be computed using Dijkstra (in time $\mathcal{O}(2^{|V|}nm(|V|+\log(nm)))$) by taking a graph with the $\mathcal{O}(2^{|V|}nm)$ nodes of the form (S,x,y).

Partial solution (60 points)

You can use the first partial solution to compute a shortest path W_v enclosing each village v. Let R_v be the region associated to W_v. Then, proceed like in the full solution, but taking $F = \bigcup_v R_v$. The running time of this algorithm is $\mathcal{O}(n^2m^2\log(nm))$.